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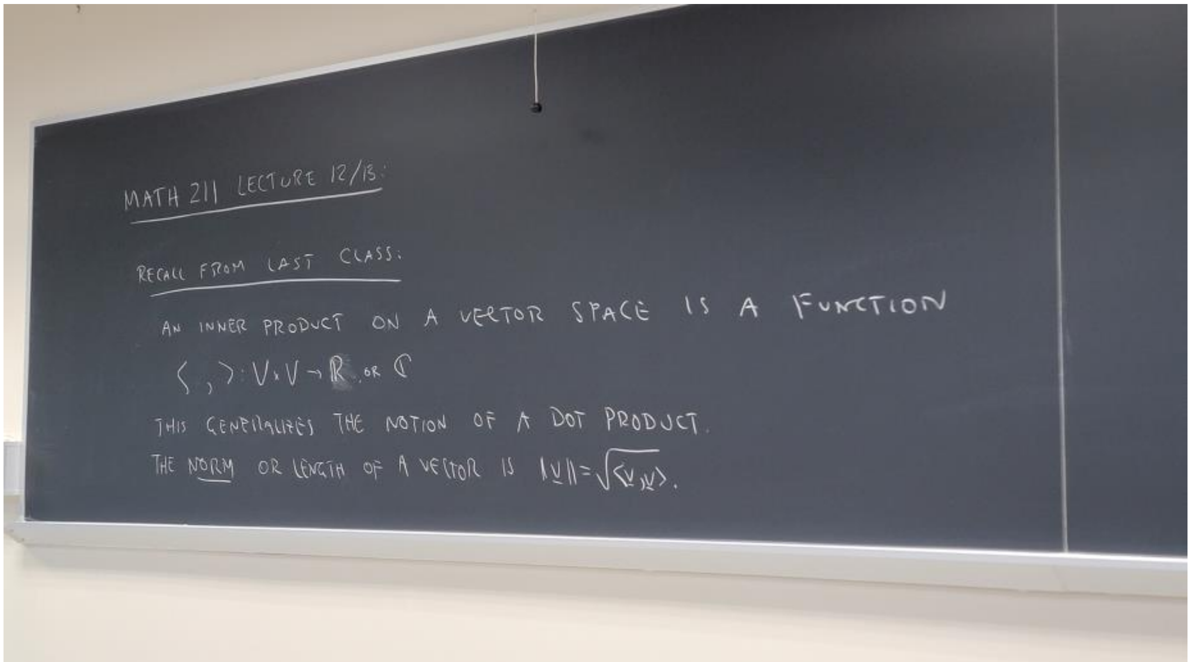
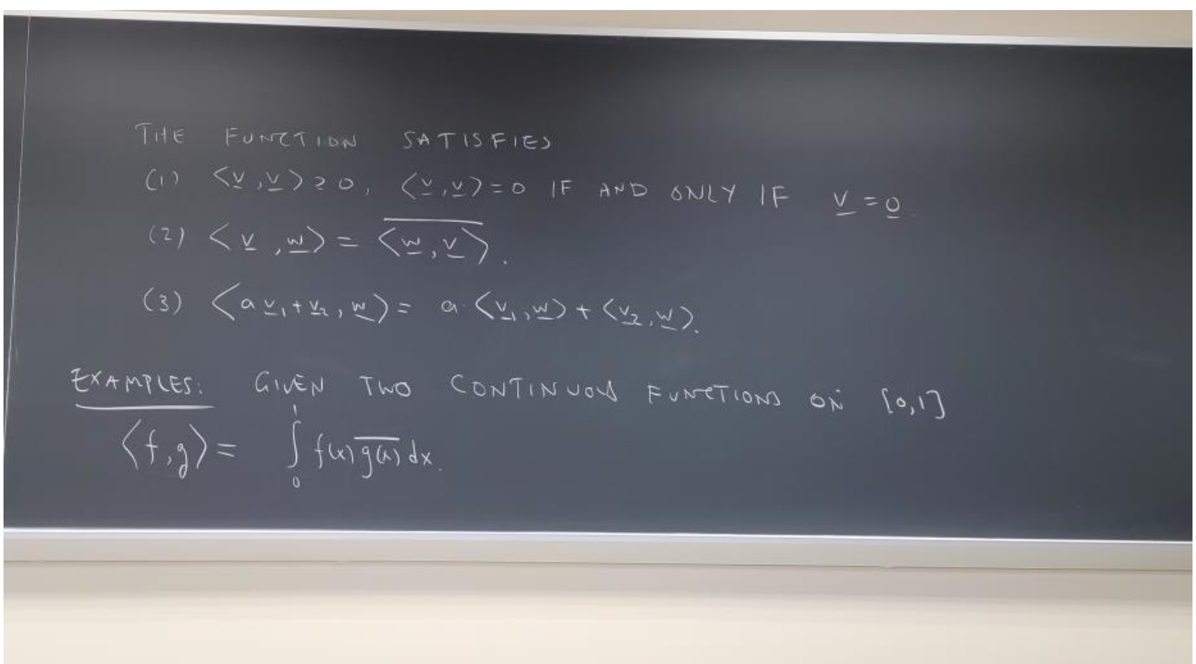
Thursday, March 9, 2023 3:44 PM





GIVEN $n \times n$ REAL MATRICES A, B,

$$\langle A, B \rangle = \mathrm{tr}\left[A^t B\right]$$

$$\langle A, A \rangle = \mathrm{tr}\begin{bmatrix} - \ \underline{v}_1^t \ - \\ - \ \underline{v}_2^t \ - \\ \vdots \\ - \ \underline{v}_n^t \ - \end{bmatrix}\begin{bmatrix} | & | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \cdots & \underline{v}_n \\ | & | & | & & | \end{bmatrix}.$$

$$= \mathrm{tr}\begin{bmatrix} \underline{v}_1 \cdot \underline{v}_1 & \underline{v}_1 \cdot \underline{v}_2 & \cdots & \underline{v}_1 \cdot \underline{v}_n \\ \underline{v}_2 \cdot \underline{v}_1 & \underline{v}_2 \cdot \underline{v}_2 & \cdots & \underline{v}_2 \cdot \underline{v}_n \\ \vdots & & & \\ \underline{v}_n \cdot \underline{v}_1 & \underline{v}_n \cdot \underline{v}_2 & \cdots & \underline{v}_n \cdot \underline{v}_n \end{bmatrix} = \|\underline{v}_1\|^2 + \|\underline{v}_2\|^2 + \cdots + \|\underline{v}_n\|^2.$$

$$= \sum_{i,j} A_{i,j}^2.$$

EXAMPLE. $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$

$$\langle A, A \rangle = 1^2 + 2^2 + 0^2 + 4^2 = 21$$

DEFINITION. THE INDUCED NORM OF AN INNER PRODUCT IS $\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$.

WE SAY $\underline{v}, \underline{w}$ ARE ORTHOGONAL IF $\langle \underline{v}, \underline{w} \rangle = 0$

(POLARIZATION IDENTITY)

THEOREM. IF $\langle \, , \rangle$ IS AN INNER PRODUCT ON A REAL VECTOR SPACE

$$\langle \underline{v}, \underline{w} \rangle = \frac{1}{4}\left(\|\underline{v}+\underline{w}\|^2 - \|\underline{v}-\underline{w}\|^2 \right).$$

IF $\langle \, , \rangle$ IS AN INNER PRODUCT IN A COMPLEX VECTOR SPACE

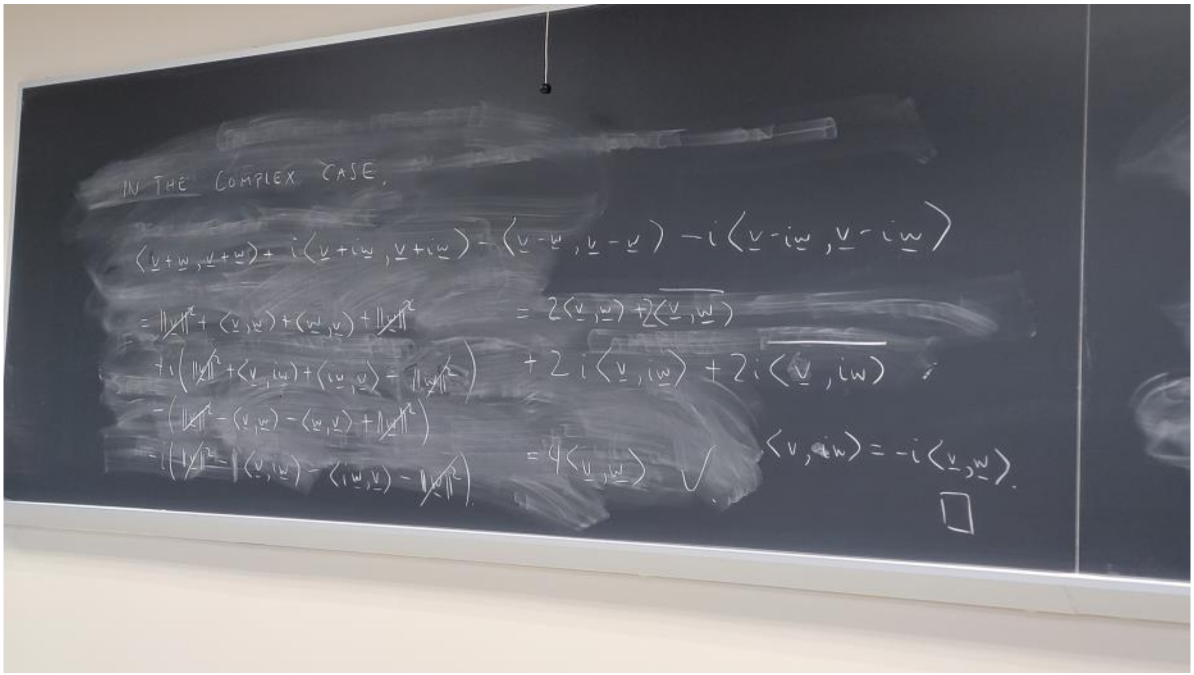
$$\langle \underline{v}, \underline{w} \rangle = \frac{1}{4}\left(\|\underline{v}+\underline{w}\|^2 + i\|\underline{v}+i\underline{w}\|^2 + \|\underline{v}-\underline{w}\|^2 - i\|\underline{v}-i\underline{w}\|^2 \right).$$

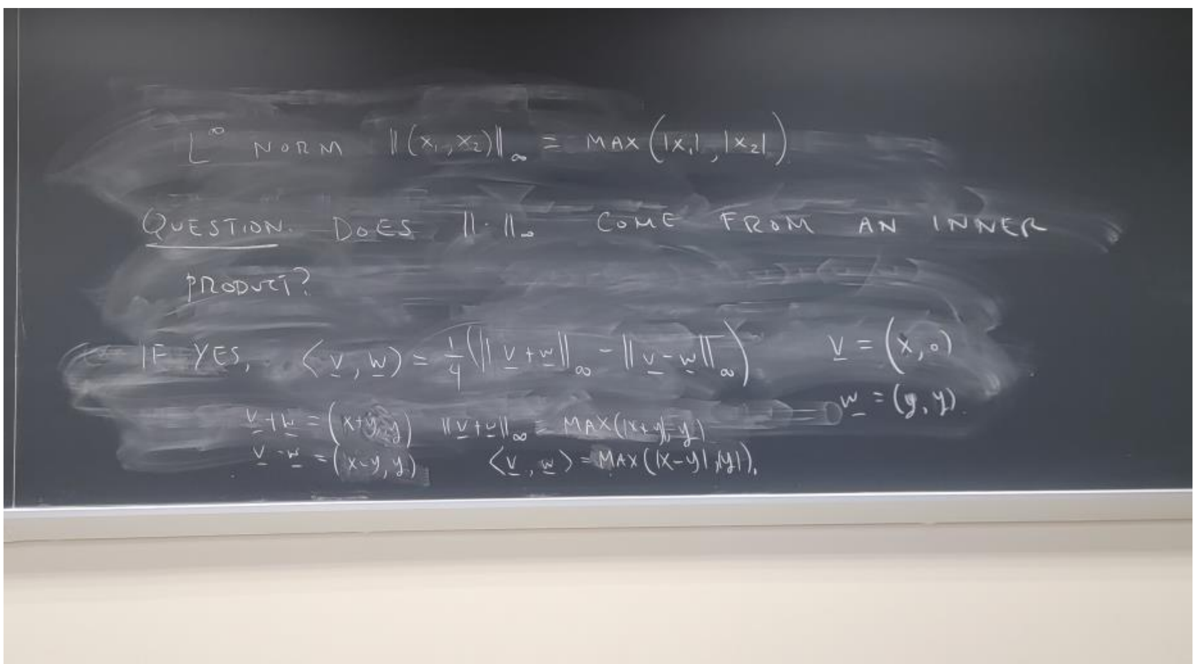
PROOF: IN THE REAL CASE,

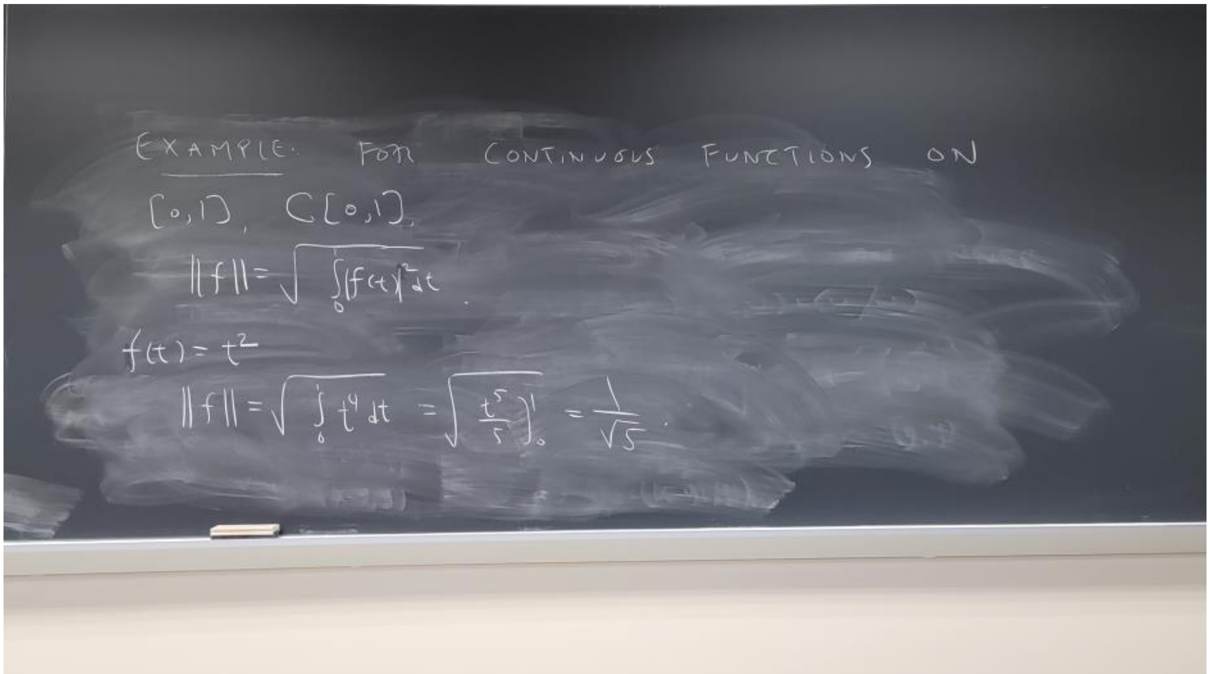
$$\|\underline{v}+\underline{w}\|^2 - \|\underline{v}-\underline{w}\|^2 = \langle \underline{v}+\underline{w}, \underline{v}+\underline{w} \rangle - \langle \underline{v}-\underline{w}, \underline{v}-\underline{w} \rangle.$$

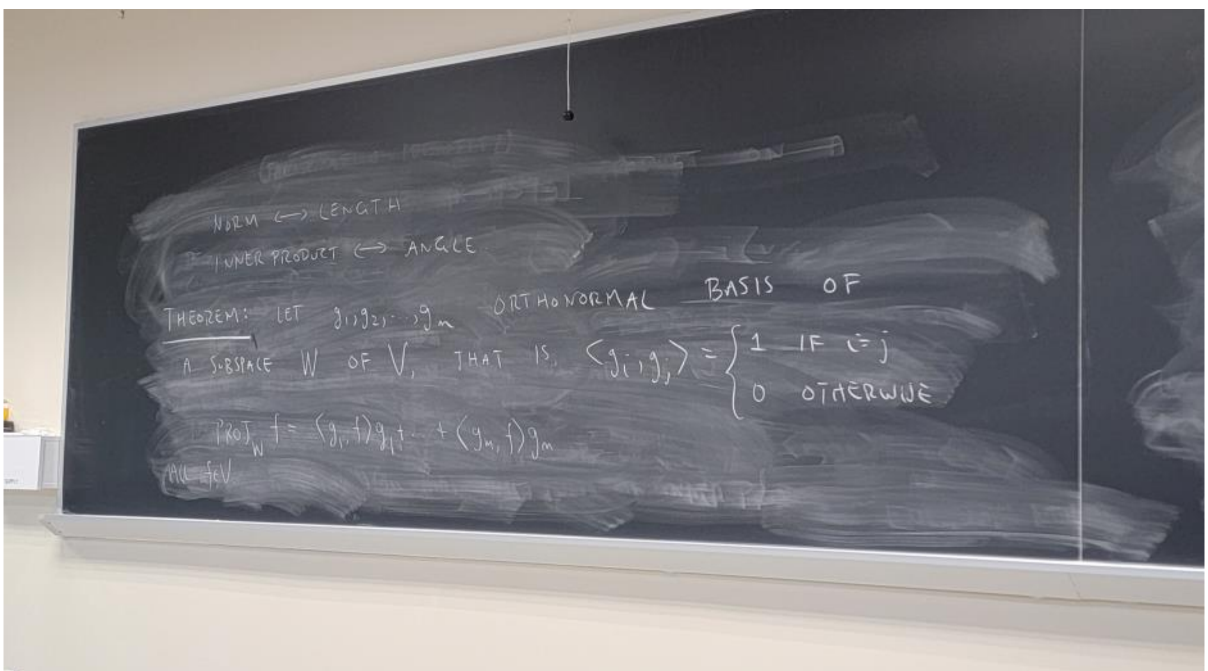
$$= \cancel{\langle \underline{v}, \underline{v} \rangle} + \langle \underline{w}, \underline{v} \rangle + \langle \underline{v}, \underline{w} \rangle + \cancel{\langle \underline{w}, \underline{w} \rangle}$$
$$- \left(\cancel{\langle \underline{v}, \underline{v} \rangle} - \langle \underline{v}, \underline{w} \rangle - \langle \underline{w}, \underline{v} \rangle + \cancel{\langle \underline{w}, \underline{w} \rangle} \right)$$

$$= 4 \langle \underline{v}, \underline{w} \rangle.$$


IN THE COMPLEX CASE.
$\langle \underline{v}+\underline{w}, \underline{v}+\underline{w}\rangle + i\langle \underline{v}+i\underline{w}, \underline{v}+i\underline{w}\rangle - \langle \underline{v}-\underline{w}, \underline{v}-\underline{w}\rangle - i\langle \underline{v}-i\underline{w}, \underline{v}-i\underline{w}\rangle$
$= \cancel{\|\underline{v}\|^2} + \langle \underline{v}, \underline{w}\rangle + \langle \underline{w}, \underline{v}\rangle + \cancel{\|\underline{w}\|^2}$
$+ i\left(\cancel{\|\underline{v}\|^2} + \langle \underline{v}, i\underline{w}\rangle + \langle i\underline{w}, \underline{v}\rangle + \cancel{\|\underline{w}\|^2}\right)$
$-\left(\cancel{\|\underline{v}\|^2} - \langle \underline{v}, \underline{w}\rangle - \langle \underline{w}, \underline{v}\rangle + \cancel{\|\underline{w}\|^2}\right)$
$-i\left(\cancel{\|\underline{v}\|^2} - \langle \underline{v}, i\underline{w}\rangle - \langle i\underline{w}, \underline{v}\rangle - \cancel{\|\underline{w}\|^2}\right)$
$= 2\langle \underline{v}, \underline{w}\rangle + 2\overline{\langle \underline{v}, \underline{w}\rangle}$
$+ 2i\langle \underline{v}, i\underline{w}\rangle + 2i\overline{\langle \underline{v}, i\underline{w}\rangle}$
$= 4\langle \underline{v}, \underline{w}\rangle$ ✓
$\langle v, iw\rangle = -i\langle \underline{v}, \underline{w}\rangle$.
$\square$


L^∞ NORM $\|(x_1, x_2)\|_\infty = \mathrm{MAX}(|x_1|, |x_2|)$
QUESTION. DOES $\|\cdot\|_\infty$ COME FROM AN INNER PRODUCT?
IF YES, $\langle \underline{v}, \underline{w}\rangle = \frac{1}{4}\left(\|\underline{v}+\underline{w}\|_\infty - \|\underline{v}-\underline{w}\|_\infty\right)$
$\underline{v} = (x, 0)$
$\underline{w} = (y, y)$
$\underline{v}+\underline{w} = (x+y, y)$ $\|\underline{v}+\underline{w}\|_\infty = \mathrm{MAX}(|x+y|, |y|)$
$\underline{v}-\underline{w} = (x-y, y)$ $\langle \underline{v}, \underline{w}\rangle = \mathrm{MAX}(|x-y|, |y|)$.


EXAMPLE. FOR CONTINUOUS FUNCTIONS ON
$[0,1]$, $C[0,1]$.
$\|f\| = \sqrt{\int_0^1 (f(t))^2 dt}$.
$f(t) = t^2$
$\|f\| = \sqrt{\int_0^1 t^4 dt} = \sqrt{\frac{t^5}{5}\Big]_0^1} = \frac{1}{\sqrt{5}}$.


NORM $\longleftrightarrow$ LENGTH
INNER PRODUCT $\longleftrightarrow$ ANGLE
THEOREM: LET $g_1, g_2, \dots, g_m$ ORTHONORMAL BASIS OF
A SUBSPACE W OF V, THAT IS, $\langle g_i, g_j \rangle = \begin{cases} 1 & \text{IF } i=j \\ 0 & \text{OTHERWISE} \end{cases}$
$\text{PROJ}_W f = (g_1, f) g_1 + \dots + (g_m, f) g_m$
ALL $f \in V$

PROOF: THE PROJECTION OF f IN W SATISFIES $f - \text{PROJ}_W f$ IS PERPENDICULAR TO W.

SO, FOR ANY $\underline{w} \in W$ $\quad \langle f - \text{PROJ}_W f, \underline{w} \rangle = 0$.

ANY $\underline{w} \in W$ MAY BE WRITTEN $\underline{w} = c_1 g_1 + \dots + c_m g_m$.

SO IT SUFFICES TO CHECK $\langle f - \text{PROJ}_W f, g_i \rangle = 0$.

$$\langle f - \text{PROJ}_W f, g_i \rangle$$

$$= \langle f, g_i \rangle - \langle (\langle f, g_1 \rangle g_1 + \dots + \langle f, g_m \rangle g_m), g_i \rangle$$

$$= \langle f, g_i \rangle - \langle f, g_i \rangle = 0. \qquad \square$$

IN APPLICATIONS, AN IMPORTANT EXAMPLE TO REMEMBER IS FOURIER SERIES.

ON $C[-\pi,\pi]$, USE INNER PRODUCT

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,g(t)\,dt.$$

$\frac{1}{\sqrt{2}}$, $\sin t$, $\cos t$, $\sin 2t$, $\cos 2t$, $\sin 3t$, $\cos 3t$, ...

ARE ORTHONORMAL.

THE <u>TRIGONOMETRIC POLYNOMIALS</u> ARE ALL COMBINATIONS OF FINITELY MANY TERMS FROM THIS LIST.

THE FOURIER COEFFICIENTS OF A FUNCTION f ARE GIVEN BY

$$a_0 = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}\,\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

$$b_n = \langle f, \cos nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt.$$

$$c_n = \langle f, \sin nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt.$$

THEOREM. IF f IS A TRIGONOMETRIC POLYNOMIAL WITH FREQUENCIES AT MOST n,

$$f(t) = a_0 + b_1 \cos t + c_1 \sin t + b_2 \cos 2t + c_2 \sin 2t + \dots + b_n \cos nt + c_n \sin nt$$

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2\,dt = a_0^2 + b_1^2 + c_1^2 + \dots + b_n^2 + c_n^2.$$

PROOF. THIS IS TRUE BECAUSE THE FUNCTIONS ARE ORTHONORMAL.

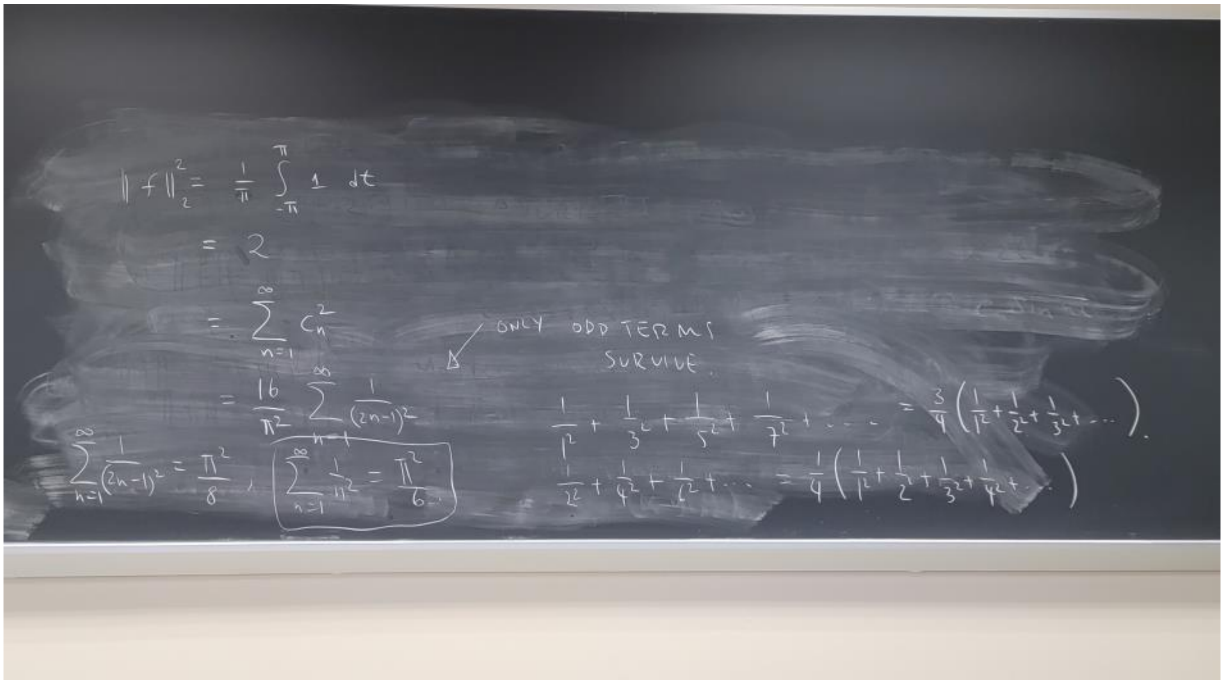
EXAMPLE:
$$f(t) = \begin{cases} -1 & -\pi \le t < 0 \\ 0 & t = 0 \\ 1 & 0 < t \le \pi \end{cases}$$

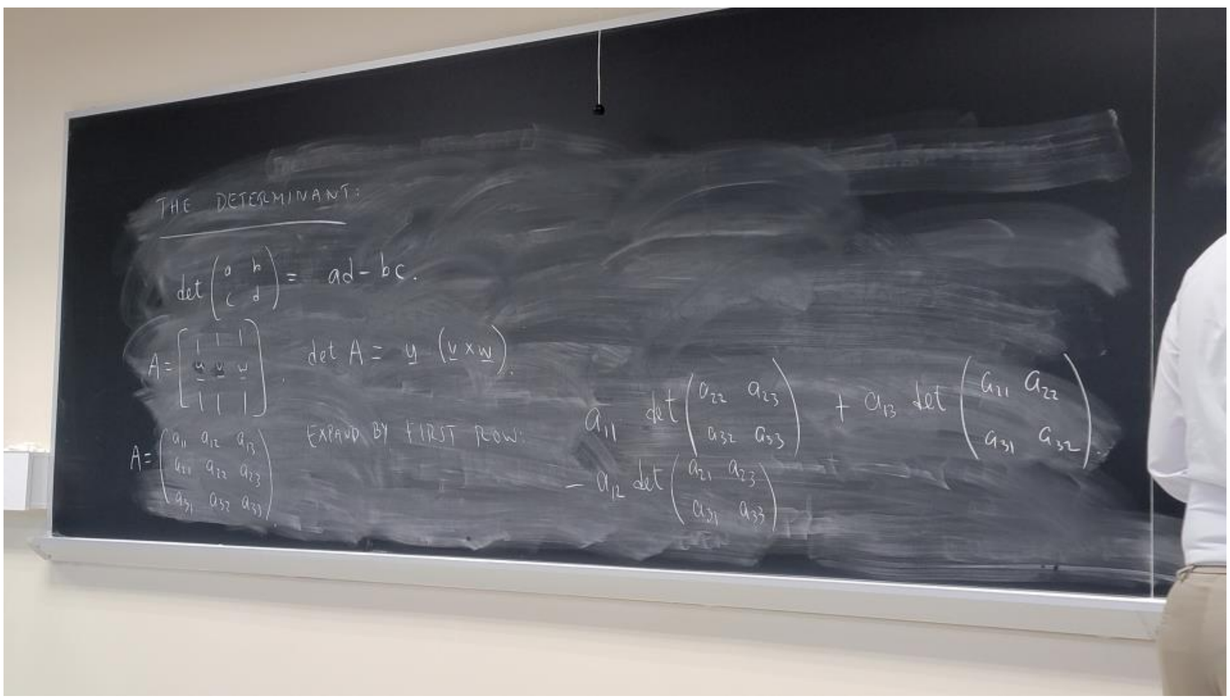
$$\left\langle f, \tfrac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dt + \frac{1}{\pi}\int_{0}^{\pi} 1\,dt = 0.$$

$$\langle f, \cos nt \rangle = \frac{-1}{\pi}\int_{-\pi}^{0} \cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \cos nt\,dt. = 0 \qquad \cos nt = \cos(-nt)$$

$$\langle f, \sin nt \rangle = -\frac{1}{\pi}\int_{-\pi}^{0} \sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt\,dt \qquad \sin nt = -\sin(-nt).$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nt\,dt = \frac{2}{n\pi}\left[-\cos nt\right]_0^{\pi} = \begin{cases} \frac{4}{n\pi} & n \text{ ODD} \\ 0 & n \text{ EVEN}. \end{cases}$$


$\|f\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \, dt$
$= 2$
$= \sum_{n=1}^{\infty} c_n^2$
ONLY ODD TERMS SURVIVE.
$= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$
$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$, $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$
$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots = \frac{3}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)$.
$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots = \frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)$


THE DETERMINANT:
$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$
$A = \begin{bmatrix} | & | & | \\ \underline{u} & \underline{v} & \underline{w} \\ | & | & | \end{bmatrix}$. $\det A = \underline{u} \cdot (\underline{v} \times \underline{w})$.
$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
EXPAND BY FIRST ROW:
$a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$.
$- a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$
$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

<u>EXAMPLE:</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ $\det A = \underbrace{\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{0 \text{ BECAUSE ROWS LIN DEP.}} + \det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

$= \det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3.$

<u>EXAMPLE:</u> IF A IS A 3×3 MATRIX

$F(A) = \det(A)$ IS A FUNCTION $\mathbb{R}^{3\times 3} \to \mathbb{R}$

<u>NOT</u> LINEAR. $\det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 8 \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

<u>DEFINITION:</u> THE DETERMINANT OF AN $n\times n$ MATRIX IS A FUNCTION

$\det\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix}$ LINEAR <u>SEPARATELY</u> IN EACH COLUMN

ALTERNATING, IN THE SENSE THAT IF YOU SWAP THE POSITION OF TWO VECTORS, THE SIGN OF THE DETERMINANT CHANGES.

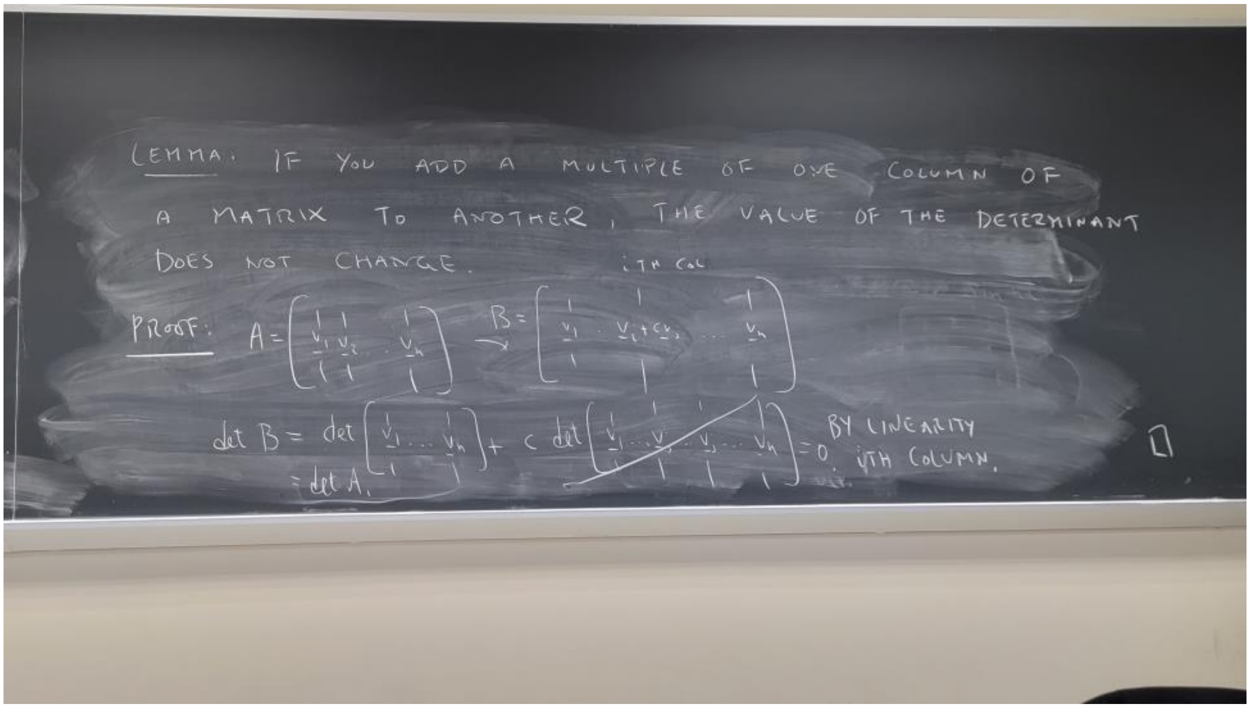
$$\det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix} = \det(I_n) = 1.$$

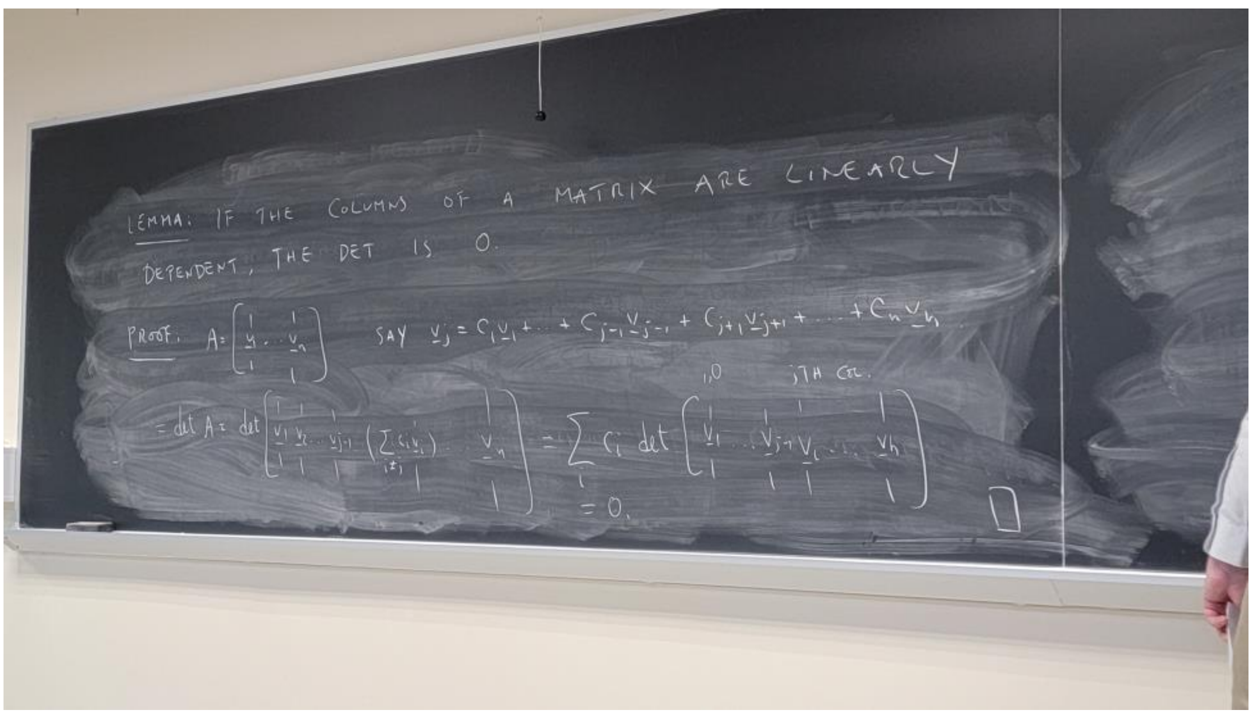
<u>REMARK.</u> THIS UNIQUELY DEFINES THE DETERMINANT.

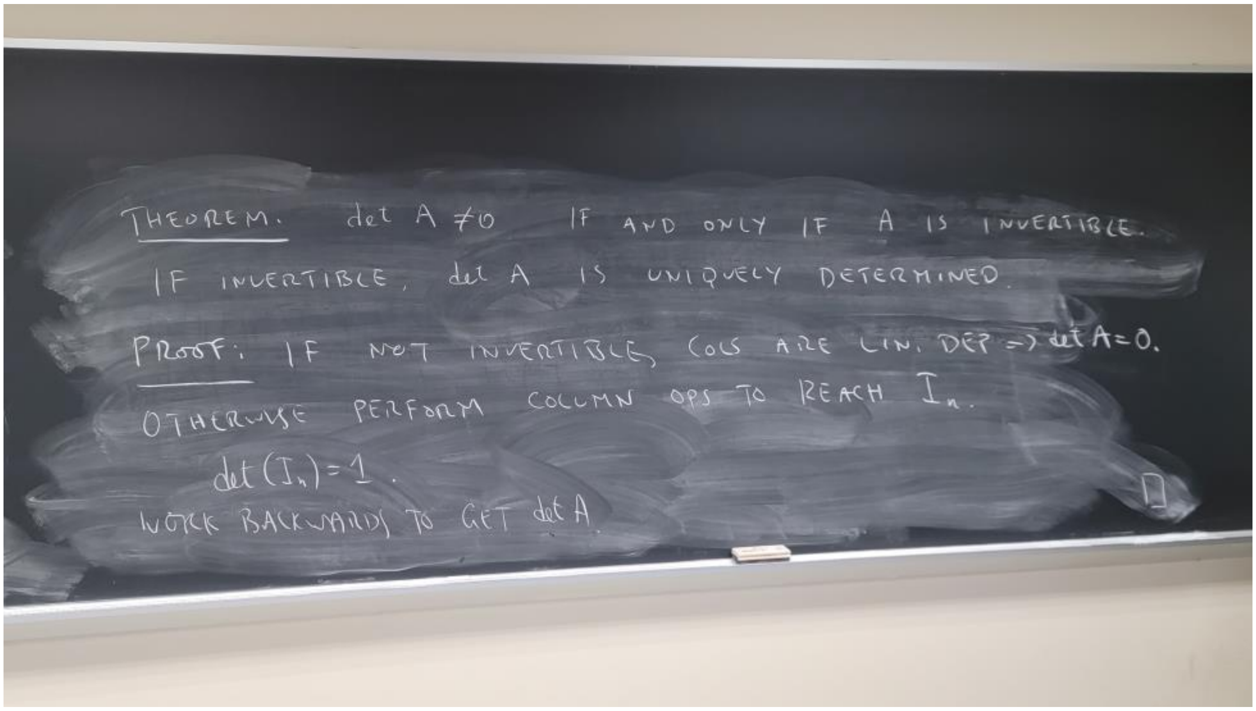
WE'LL SHOW THERE EXISTS SUCH A FUNCTION BY CONSTRUCTION.
FIRST WE'LL SHOW THERE CAN BE ONLY ONE SUCH FUNCTION.

<u>LEMMA</u> IF TWO COLUMNS OF A MATRIX ARE EQUAL, THE DETERMINANT IS ZERO.

<u>PROOF</u> $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$. IF $v_i = v_j$, $i<j$. SWAP POSITIONS WITHOUT CHANGING THE MATRIX.
$\Rightarrow \det A = -\det A \Rightarrow \det A = 0$.


LEMMA: IF YOU ADD A MULTIPLE OF ONE COLUMN OF A MATRIX TO ANOTHER, THE VALUE OF THE DETERMINANT DOES NOT CHANGE.
PROOF:
iTH COL
BY LINEARITY
jTH COLUMN.
= det A.
= 0.


LEMMA: IF THE COLUMNS OF A MATRIX ARE LINEARLY DEPENDENT, THE DET IS 0.
PROOF.
SAY
jTH COL.
= 0.


THEOREM. det A $\neq 0$ IF AND ONLY IF A IS INVERTIBLE.
IF INVERTIBLE, det A IS UNIQUELY DETERMINED.
PROOF: IF NOT INVERTIBLE, COLS ARE LIN. DEP $\Rightarrow$ det A = 0.
OTHERWISE PERFORM COLUMN OPS TO REACH I_n.
$\det(I_n) = 1$.
WORK BACKWARDS TO GET det A.
□